Chapter 7

MECHANICS OF BEDLOAD TRANSPORT

7.1 INTRODUCTION

In the following series of lectures, a relation for bedload transport is sought. In order to simplify the problem, the flow is taken to be steady, uniform, and fully turbulent, obeying the rough logarithmic law in a range near the bed where bedload transport takes place. The relation sought is of the form \( q = f(\tau_s) \) as outlined in Chapter 5. The analysis can be generalized to the case for which the flow field near the bed is nonuniform in the \( s \) and \( n \) directions, and varying in time.

The effect of turbulence on particle motion is taken to be negligible, in accordance with the assumption \( u_s/v_s < 1 \).

7.2 EQUATIONS OF PARTICLE MOTION

The appropriate equations governing particle motion were presented in Chapter 4. For simplicity, the \( z \)-coordinate is taken to be approximately upward vertical. Variation in the \( n \)-coordinate is taken to be negligible. The mean near-bed flow field is given by

\[
\bar{u}(s, n, z, t) = 2.5u_s \ln \left( \frac{30z}{k_s} \right) \quad (7.1a)
\]

\[
\bar{v}(s, n, z, t) = \bar{w}(s, n, z, t) = 0 \quad (7.1b)
\]
The equations of particle motion in Chapter 4 thus take the form

\[(\rho_s + c_{m\rho}) V_p \frac{d u_p}{d t} = -\frac{1}{2} \rho c_D A_p |u_r| (u_p - u_f) + \rho (1 + c_m) V_p \frac{d u_f}{d t}\] \hspace{1cm} (7.2a)

\[(\rho_s + c_{m\rho}) V_p \frac{d w_p}{d t} = -\frac{1}{2} \rho c_D A_p |u_r| w_p - (\rho_s - \rho) g V_p + \frac{1}{2} \rho c_L A_p \left( u_{rT}^2 - u_{rB}^2 \right)\] \hspace{1cm} (7.2b)

In the above relations,

\[u_f(t) \equiv \bar{u}(s_p(t), n_p(t), z_p(t), t) = 2.5 u_s \ln \left( \frac{30 z_p(t)}{k_s} \right)\] \hspace{1cm} (7.3)

denotes the fluid velocity extrapolated to the particle centroid, and \(|u_r|\) denotes the magnitude of the particle velocity vector relative to the fluid, given by

\[|u_r|^2 = (u_p - u_f)^2 + w_p^2\] \hspace{1cm} (7.4a)

Furthermore, in the right term in (7.2b), \(u_{rT}\) and \(u_{rB}\) represent the magnitude of the relative particle velocity at the top and bottom of the particle, respectively; i.e.

\[u_{rT} = (u_p - u_{fT})^2 + w_p^2, \quad u_{rB} = (u_p - u_{fB})^2 + w_p^2\] \hspace{1cm} (7.4b, c)

where

\[u_{fT} = 2.5 u_s \ln \left( 30 z_p + \frac{1}{2} D \right), \quad u_{fB} = 2.5 u_s \ln \left( 30 z_p - \frac{1}{2} D \right)\] \hspace{1cm} (7.4d, e)

In equation 7.2a, the fluid acceleration term \(d u_f/dt\) must be evaluated following the particle; that is, from equation 7.3,

\[\frac{d u_f}{d t} = \frac{\partial u_f}{\partial t} + u_p \frac{\partial u_f}{\partial s_p} + w_p \frac{\partial u_f}{\partial z_p} = 2.5 w_p \frac{u_s}{z_p}\] \hspace{1cm} (7.4f)

### 7.3 DYNAMIC FRICTION COEFFICIENT

Each time a saltating particle strikes the bed, it transfers a portion of its forward momentum to the bed. The net result of successive collisions is a mean streamwise shear force exerted on the bed by obliquely-striking particles. This force is now computed.

Because the collision is both inelastic and oblique, the forward momentum of the particle just after collision is less than the value just before collision. That is, the parameter \(\Delta u_p > 0\), where

\[\Delta u_p = u_p|_{in} - u_p|_{out}\] \hspace{1cm} (7.5a)
(In the case of a stochastic treatment, $\Delta u_p$ must be based on an ensemble average over many saltations.) The momentum transferred from the particle to the bed per collision is thus given by $\rho_s V_p \Delta u_p$. On the average, there is a collision with the bed each $\lambda_s/\bar{u}_p$ seconds. The streamwise shear force $F_{gr}$ (the subscript being short for “grain”) exerted on the bed due to collision is thus given by the momentum transfer per unit time

$$F_{gr} = \rho_s V_p \Delta u_p \frac{\bar{u}_p}{\lambda_s}$$

(7.5b)

By Newton’s third law, this forward shear force of the saltating particles on the bed corresponds to a resistive force of the bed on the moving grains of equal magnitude and opposite direction.

It is now possible to define a coefficient of dynamic Coulomb friction $\mu_d$ in analogy to the coefficient of static Coulomb friction introduced previously:

$$\mu_d = \frac{\text{tangential resistive force of collision}}{\text{submerged particle weight}} = \frac{F_{gr}}{(\rho_s - \rho)gV_p} = \frac{R + 1}{R} \frac{\Delta u_p \bar{u}_p}{g\lambda_s}$$

(7.6)

The dynamic friction coefficient $\mu_d$ can in general be represented in the following form:

$$\mu_d = \mu_d (\tau^*, R_{ep}, R)$$

(7.7)

### 7.4 SOME CONTINUITY RELATIONS FOR SALTATING GRAINS

At this point, it is useful to generalize from the case of a single saltating grain to a collection of grains moving over, and exchanging with an immobile bed of similar grains. For simplicity all the saltation trajectories are taken to be identical, although the analysis readily generalizes to the stochastic case. From this point on, the treatment is fundamentally Eulerian rather than Lagrangian.

Consider the illustration below.
Let $E_s$ denote the volume upward flux of grains from the bed, due to erosion of previously immobile particles or ejection of colliding particles. Furthermore, let $C_u(z)$ denote the volume concentration of upward-moving ($w_p > 0$) particles at level $z$, $C_d(z)$ denote the corresponding concentration of downward-moving particles, and $C_s(z)$ denote the total volume concentration of saltating bedload particles.

The velocities $u_{pu}(z)$ and $u_{pd}(z)$ now correspond to the Eulerian streamwise grain velocities for upward- and downward-moving particles at elevation $z$; the corresponding upward normal Eulerian grain velocities are $w_{pu}$ and $w_{pd}$.

The following conditions hold from continuity:

$$C_u w_{pu} = -C_d w_{pd} = E_s \quad (7.8)$$

$$C_s = C_u + C_d \quad (7.9)$$

Furthermore, the phase-averaged (upward- and downward-moving) Eulerian streamwise velocity of saltating grains $u_{se}$ is given by

$$u_{se} = \left( C_u u_{pu} + C_d u_{pd} \right) / C_s \quad (7.10)$$

The volume streamwise bedload transport rate $q$ is given as

$$q = \int_{h_s}^{h_{bed}} C_s(z) u_{se} \, dz \equiv \bar{q} \bar{u}_{se} \quad (7.11a)$$

Here $h_s$ is the thickness of the bedload layer, and

$$\bar{q} = \int_{h_s}^{h_s} C_s(z) \, dz \quad (7.11b)$$

denotes the volume of moving grains per unit bed area, and $\bar{u}_{se}$ is a flux-averaged value of $u_{se}$ given by

$$\bar{u}_{se} = \frac{\int C_s u_{se} \, dz}{\int C_s \, dz} \quad (7.11c)$$

Finally, the following continuity relation holds:

$$q = E_s \lambda_s \quad (7.12)$$
It is now possible to compute the net upward flux of streamwise momentum $F_{gmsz}$ due to saltating grains: this quantity is given by

$$F_{gmsz} = \rho_s \left( C_u u_{pw} + C_d u_{pd} w_{pd} \right)$$  \hspace{1cm} (7.13)

In analogy to the Reynolds stress, the shear grain stress $\tau_g$, or the shear stress associated with the upward normal transfer of streamwise momentum by saltating grains, is given by

$$\tau_g = -F_{gmsz} = -\rho_s \left( C_u u_{pw} + C_d u_{pd} w_{pd} \right)$$ \hspace{1cm} (7.14a)

Reducing the above relation with the aid of equations (7.8), (7.11), and (7.12), it is found that

$$\tau_g(z) = \rho_s \xi \frac{u_{pd} - u_{pu}}{\lambda_s}$$ \hspace{1cm} (7.14b)

This relation is in precise analogy to equation (7.5), except that it holds at an arbitrary distance above the bed, as well as at the bed itself.

Bagnold (1957) used some of the above ideas to gain a picture of the interaction of the flow and saltating grains. Consider the diagram below, corresponding to equilibrium flow in a wide rectangular channel with a mobile bed. Where $H$ denotes flow depth, it is assumed that $D/H << 1$. The fluid shear stress $\tau$ varies linearly in depth right up to the top of the saltation layer $z = h_s$. As long as saltation is confined to a very thin layer compared to the depth, the fluid shear stress $\tau_T$ at the top of the saltation layer is given accurately by $\rho u_s^2$, where

$$\tau_T = \rho u_s^2 = \rho g H S$$ \hspace{1cm} (7.15)

For distances of the order of the grain size above $z = h_s$, $\tau$ is very nearly equal to $\tau_T$, in accordance with the constant-stress approximation of Chapter 3 (Page 29).
Now the essential role of saltating grains as regards momentum balance is to effect a transfer of streamwise momentum from the fluid phase to the solid phase. On the average, the saltating grains move more slowly than the surrounding fluid. As a result, they extract momentum from the fluid via drag. This streamwise momentum is then fluxed toward the bed, giving rise to a grain shear stress \( \tau_g \). The value of the grain shear stress at the bed, \( \tau_{gb} \), corresponds to the force per unit area exerted by the grains on the bed.

Within the bedload layer, any net gain of momentum by the grains must correspond to a net loss by the fluid. It thus follows that

\[
\tau + \tau_g = \tau_T
\]

That \( \tau_g \) is indeed positive in the bedload layer (corresponding to a net downward flux of streamwise momentum) can be seen from equation 7.14b; in general, \( u_{pd} \) should exceed \( u_{pu} \), because downward-moving particles have been subject to the accelerative effect of the fluid than (recently effected) upward particles.

It is thus seen that within the bedload layer the fluid stress \( \tau \) is reduced to the value \( \tau_T - \tau_g \). The larger the number of particles participating in bedload motion, the larger is the value of the volume in transport per unit bed area \( \xi \); it is clear from equation (7.14) that this implies a larger value of \( \tau_g \) and a further reduction in fluid stress \( \tau \).

Bagnold hypothesized that this process is limited by the critical boundary shear stress \( \tau_{bc} \). That is, more and more particles should be entrained into bedload motion until such point as the boundary fluid stress \( \tau_b \) drops to the value \( \tau_{bc} \); beyond this point there is no more net entrainment of particles, and an equilibrium state of bedload transport is reached. Thus, from equations (7.5) and (7.14)–(7.15), it follows that

\[
\tau_{gb} = \rho u_e^2 - \tau_{bc}
\]

where \( \tau_{gb} \) is given by

\[
\tau_{gb} = \rho_s \xi u_{se} \frac{\Delta u_p}{\lambda_s}
\]

Here \( \Delta u_p \) is evaluated at the bed in accordance with equation (7.5a).

Further progress can be made by introducing the coefficient of dynamic Coulomb friction \( \mu_d \). In the context of the present Eulerian analysis, this can be defined to be the ratio of \( \tau_{gb} \) to the submerged weight of the moving bedload particles per unit bed area: that is

\[
\mu_d = \frac{\tau_{gb}}{\rho R g \xi}
\]

Reducing with the aid of equation (7.17b), it is found that

\[
\mu_d = \frac{R + 1}{R} \frac{\Delta u_p u_{se}}{g \lambda_s}
\]
The Eulerian definition for $\mu_d$ given in equation (7.19) is seen to be identical to the Lagrangian definition of equation (7.6), to the extent that the mean Eulerian bedload velocity $\bar{u}_{se}$ can be equated to the mean Lagrangian particle velocity $\bar{u}_p$. This assumption is made here, allowing for the use of a relation of the form of equation (7.7), determined from a Lagrangian analysis of grain saltation.

Between equations (7.17a), (7.18), and (7.19), then, the following relation for bedload concentration per unit bed area $\xi$ is obtained:

$$\mu_d \rho R g \xi = \rho u_{se}^2 - \tau_{bc}$$  \hspace{1cm} (7.20a)

This relation can be placed in dimensionless form by dividing by $\rho R g D$, yielding

$$\frac{\xi}{D} = \frac{\tau^* - \tau_c^*}{\mu_d}$$  \hspace{1cm} (7.20b)

### 7.6 BEDLOAD TRANSPORT RELATION

It is now possible to specify the form of the relation for bedload transport. Between equations 7.7, 7.11a, and the identification of $\bar{u}_{se}$ and $\bar{u}_p$, it is found that

$$q^* = \tau^{*1/2} (\tau^* - \tau_c^*) \frac{f_p(\tau^*, R_{ep}, R)}{\mu_d (\tau^*, R_{ep}, R)}$$  \hspace{1cm} (7.21)

Here $q^*$ is a dimensionless bedload transport rate known as the Einstein number, first introduced by H. A. Einstein in 1950, and given by

$$q^* = \frac{q}{\sqrt{R g D D}}$$  \hspace{1cm} (7.22)

To date, only two research groups have carried out complete derivations of the bedload function in water, including the stochastic element associated with the splash function. They are Wiberg and Smith (1989), and Sekine (1989). The references are given at the end of the previous chapter.
7.7 MACROSCOPIC ANALYSIS OF BEDLOAD TRANSPORT: ASHIDA-MICHIUE

Ashida and Michiue (1972) have presented a simplified analysis of bedload transport that cuts through the complexity of saltation. A nearly identical analysis was offered several years later by Engelund and Fredsøe (1976). The treatment is of sufficient value to warrant reproduction herein.

The authors neglect the fluid acceleration term in equation 7.2a. That equation is valid only while the particle is actually saltation. They average particle motion over many saltations, which requires including the momentum loss associated with bed collision. Thus between equations 7.2a, 7.5b, and 7.6, it is found that

\[
\frac{d\bar{u}_p}{dt} = \frac{1}{2} \rho_c D_A |\bar{u}_r| (\bar{u}_p - \bar{u}_f) - \mu_d \rho R g V_p \tag{7.23}
\]

Here the overbar denotes an average over many saltations. It is assumed that \( \bar{d} \) takes a constant value of 0.5. While this is only a crude approximation, it is not entirely unreasonable, as evidenced by the rather weak variation in \( \bar{d} \) in Figure 2.

Under conditions of equilibrium saltation, the particle acceleration term \( \bar{u}_p \) vanishes. Ashida and Michiue further drop the term \( w_p \) in evaluating \( |\bar{u}_r| \) from equation 7.4a, and then make the rather bold assumption (and formally unjustified) assumption of pulling the average inside the nonlinear drag terms:

\[
|\bar{u}_r| (\bar{u}_p - \bar{u}_f) = -\left[ (\bar{u}_p - \bar{u}_f)^2 + w_p^2 \right]^{1/2} (\bar{u}_f - \bar{u}_p) \approx - (\bar{u}_f - \bar{u}_p)^2 \tag{7.24}
\]

Under the stated assumptions, then equation 7.23 and 7.24 can be solved for \( \bar{u}_p \):

\[
\bar{u}_p = \bar{u}_f - \left( \frac{4}{3} \frac{\mu_d}{c_D} \frac{R g D}{c_D} \right)^{1/2} \tag{7.25}
\]

Ashida and Michiue estimate \( \bar{u}_f \) to be a constant, evaluated from the logarithmic law at \( z = k_s \); from equation 7.3, then,

\[
\frac{\bar{u}_f}{u_*} = 8.5 \tag{7.26}
\]

It is seen from Figure 2 that this assumption is reasonable. Between equations 7.24 and 7.25, it is possible to derive a relation for the threshold value of \( u_* \) from the condition \( \bar{u}_p = 0 \):

\[
u_{sc} = \frac{1}{8.5} \left( \frac{4}{3} \frac{\mu_d}{c_D} \frac{R g D}{c_D} \right)^{1/2} \tag{7.27a}
\]
7.7. MACROSCOPIC ANALYSIS OF BEDLOAD TRANSPORT: ASHIDA-MICHIUE

or in terms of critical Shields stress,

\[ \tau_c^* = \frac{4}{3} \frac{\mu_d}{c_D F^2} \]  

(7.27b)

where \( F = 8.5 \). Equation 7.27b can be taken to be crudely analogous to equation 6.11. The former corresponds to conditions at which a moving grain stops, whereas the latter pertains to conditions at which an immobile grain begins movement.

Substituting equation 7.26 and 7.27a into equation 7.25, the following relation is found for mean streamwise particle velocity:

\[ \frac{\bar{u}_p}{u_*} = 8.5 \left[ 1 - \left( \frac{\tau_c^*}{\tau_c^{*1/2}} \right)^{1/2} \right] \]  

(7.28a)

or after some manipulation

\[ \frac{\bar{u}_p}{\sqrt{RgD}} = 8.5 \left( \tau^{1/2} - \tau_c^{*1/2} \right) \]  

(7.28b)

Equations 7.11a and 7.20b are used to complete the analysis; with the assumed value of \( \mu_d \) of 0.5, the following final form for bedload transport is obtained:

\[ q^* = 17 \left( \tau^* - \tau_c^* \right) \left( \tau^{*1/2} - \tau_c^{*1/2} \right) \]  

(7.29)

Ashida and Michiue recommend a value of \( \tau_c^* \) of 0.05 in their relation. It was verified with uniform material ranging in size from 0.3 mm to 7 mm. A test of their relation against data is shown in the following figure:
7.8 A SMORGASBORD OF BEDLOAD TRANSPORT RELATIONS

A large number of bedload relations can be expressed in the general form

\[ q^* = q^* (\tau^*, R_{cp}, R) \]  \hspace{1cm} (7.30)

In addition to the relations of Ashida and Michiuie, the following relations are of interest.

Meyer-Peter and Muller (1948):

\[ q^* = 8 (\tau^* - \tau_c^*)^{3/2} \]  \hspace{1cm} (7.31)

where \( \tau_c^* = 0.047 \). This formula is empirical in nature; it was verified with data for uniform gravel.

Engelund and Fredsøe (1976):

\[ q^* = 18.74 (\tau^* - \tau_c^*) \left( \tau^{*1/2} - 0.7\tau_c^{*1/2} \right) \]  \hspace{1cm} (7.32)
where $\tau_c^* = 0.05$. This formula resembles that of Ashida and Michiue because the derivation is almost identical.

Fernandez Luque and van Beek (1976):

$$q^* = 5.7 \left(\tau^* - \tau_c^*\right)^{3/2}$$

where $\tau_c^*$ varies from 0.05 for 0.9 mm material to 0.058 for 3.3 mm material. The relation is empirical in nature.

Wilson (1966):

$$q^* = 12 \left(\tau^* - \tau_c^*\right)^{3/2}$$

where $\tau_c^* = 0.??$. This relation is empirical in nature; most of the data used to fit it pertain to very high rates of bedload transport.

Einstein (1950):

$$q^* = q^* (\tau^*)$$

where the functionality is implicitly defined by the relation

$$1 - \frac{1}{\sqrt{\pi}} \int_{-(0.143/\tau^*)^{-2}}^{(0.143/\tau^*)^{-2}} e^{-t^2} dt = \frac{43.5q^*}{1 + 43.5q^*}$$

Note that this relation contains no critical stress. It has been used for uniform sand gravel.

Yalin (1963):

$$q^* = 0.635s\tau^*^{1/2} \left[1 - \frac{\ln(1 + a_2s)}{a_2s}\right]$$

where

$$a_2 = 2.45 (R + 1)^{0.4} \tau_c^{*1/2}; \quad s = \frac{\tau^* - \tau_c^*}{\tau_c^*}$$

and $\tau_c^*$ is evaluated from a standard Shields curve. Two constants in this formula were evaluated with the aid of data quoted by Einstein (1950), pertaining to 0.8 mm and 28.6 mm material.

Several of these relations are plotted in the following figure:
They tend to be rather similar in nature. Scores of similar relations could be quoted.

REFERENCES


- Fernandez Luque, R., and van Beek, R., Erosion and transport of bed sediment, J. Hydraulic Research, 14(2), 1976, 127-144.

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